Classical solutions for a logarithmic fractional diffusion equation

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Abstract

We prove global existence and uniqueness of strong solutions to the logarithmic porous medium type equation with fractional diffusion

$$\partial_t u + (-\Delta)^{1/2} \log(1+u) = 0,$$

posed for $x \in \mathbb{R}$, with nonnegative initial data in some function space of $L \log L$ type. The solutions are shown to become bounded and C^∞ smooth in (x,t) for all positive times. We also reformulate this equation as a transport equation with nonlocal velocity and critical viscosity, a topic of current relevance. Interesting functional inequalities are involved.

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1 Introduction

In this paper we develop the basic existence, uniqueness and regularity theory for the problem

$$\begin{cases} \partial_t u + (-\Delta)^{1/2} \log(1+u) = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x) \geq 0, & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

The equation in (1.1) can be viewed as the limit $m \to 0$ in the so-called fractional porous medium equation,

$$\partial_t u + (-\Delta)^{\sigma/2} u^m = 0, \qquad m > 0, \quad 0 < \sigma < 2, \tag{1.2}$$

after a shift in the u-variable and a change in the time scale. The latter equation was treated in our papers [19], [20], where it was proved that it generates a contraction semigroup in $L^1(\mathbb{R}^N)$ for any dimension $N \geq 1$, and that solutions become instantaneously bounded and C^α in space and time for data in $L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ with $p \geq 1$ larger than a critical value $p_* = N(1-m)/\sigma$.

The difficulty we face here is that, according to those papers, the logarithmic diffusion is borderline for regularity questions when $\sigma = 1$ and $N = 1$ for data in the natural space $L^1(\mathbb{R})$. This entails a very delicate critical-case analysis and a new type of regularity results: besides the expected result for $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ with $p > 1$, we obtain that solutions become immediately bounded when f belongs to an $L \log L$ space, almost $L^1(\mathbb{R})$ but not quite. This also offers some novelty when compared to the existing results for the standard porous medium equation, given by (1.2) with $\sigma = 2$, which are gathered in [31] and [32]. Actually, we go on to prove that the solutions are C^∞ in space and time, hence classical.

Let us remark that the method proposed to tackle regularity has a more general scope. Actually, it can be applied to positive solutions of equations of the form $\partial_t u + (-\Delta)^{\sigma/2}\varphi(u) = 0$ posed in $\mathbb{R}^N$ under quite unrestrictive assumptions on the nonlinearity. We will study this issue in a forthcoming work.

A further motivation for our study comes from the following connection: equation (1.1) can be transformed through a special nonlocal change of variables of the Bäcklund type into the transport equation

$$\partial_\tau v - \widetilde{H}(v)\partial_y v + \partial_y \widetilde{H}(v) = 0, \qquad y \in \mathbb{R}, \quad \tau > 0, \tag{1.3}$$

where $\widetilde{H}$ stands for a nonlocal operator which is a modification of the Hilbert transform. If instead of $\widetilde{H}$ we had the standard Hilbert transform H, using the identity $\partial_y H = (-\Delta)^{1/2}$, valid when these operators are applied to regular functions, we would get

$$\partial_\tau u - H(v)\,\partial_y v + (-\Delta)^{1/2} v = 0, \tag{1.4}$$

which is the transport equation with fractional diffusivity proposed by Córdoba, Córdoba and Fontelos in [10]. This is one of the several one-dimensional models considered in the last years to recast the main properties of the three-dimensional incompressible Euler equation and the two-dimensional quasigeostrophic equation, beginning with the work by Constantin, Lax and Majda [9].

Conveniently reformulated, our results for problem (1.1) produce existence and uniqueness of a *classical* global in time solution for equation (1.3) for all initial data in $L^1(\mathbb{R})$. This is a remarkable variation with respect to the results available for problem (1.4): a global in time solution is only known to exist if the initial value belongs to the Sobolev space of fractional order $H^{1/2}(\mathbb{R})$, in which case it is in $H^1(\mathbb{R})$ for almost every $t > 0$; see Dong [13]. From this regularity one might try to use the techniques of Kiselev, Nazarov and Shterenberg [15] to obtain further smoothness in space.

The application of the present approach to the transport equation (1.4) is not immediate and needs further study. We believe that the connection between fractional diffusion and nonlocal transport problems is worth pursuing, since it may lead to the fruitful combination of very different techniques.

As said above, the case $\sigma = N = 1$ is critical in various aspects, in particular with respect to Sobolev embeddings. Thus, in the course of the proof of the smoothing effect in the mentioned $L \log L$ space we need to use a critical fractional Trudinger type embedding due to Strichartz; see [27]. We generalize this embedding to other values of the exponents, a result in pure functional analysis that we hope could be of further application.

2 Preliminaries and main results

We recall that the nonlocal operator $(-\Delta)^{\sigma/2}$, $\sigma \in (0,2)$, is defined for any function $g : \mathbb{R}^N \to \mathbb{R}^N$ in the Schwartz class through the Fourier transform,

$$\mathcal{F}\left((-\Delta)^{\sigma/2} g\right)(\xi) = |\xi|^{\sigma} \mathcal{F}(g)(\xi),$$

or via the (hypersingular) Riesz potential,

$$(-\Delta)^{\sigma/2} g(x) = C_{N,\sigma} \text{ P.V.} \int_{\mathbb{R}} \frac{g(x) - g(y)}{|x-y|^{N+\sigma}}\, dy, \tag{2.1}$$

where $C_{N,\sigma}$ is a normalization constant; see for example [16]. In our case, $N = \sigma = 1$, the constant is $C_{1,1} = 1/\pi$, and we also have $(-\Delta)^{1/2} = H\partial_x$, where H denotes the Hilbert transform operator, defined trough

$$Hf(x) = \frac{1}{\pi} \text{ P.V.} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy.$$

If we multiply the equation in (1.1) by a test function φ and "integrate by parts", we obtain

$$\int_0^\infty \int_{\mathbb{R}} u\, \partial_t \varphi \, dx dt - \int_0^\infty \int_{\mathbb{R}} (-\Delta)^{1/4} \log(1+u) (-\Delta)^{1/4} \varphi \, dx dt = 0. \tag{2.2}$$

This identity will be the basis of our definition of a weak solution. The integrals in (2.2) make sense if u and $\log(1+u)$ belong to suitable spaces. The correct space for $\log(1+u)(\cdot,t)$ is the homogeneous fractional Sobolev space $\dot{H}^{1/2}(\mathbb{R})$, defined as the completion of $C_0^\infty(\mathbb{R})$ with the norm

$$\|\psi\|_{\dot{H}^{1/2}} = \left(\int_{\mathbb{R}} |\xi| |\hat{\psi}|^2 \, d\xi\right)^{1/2} = \|(-\Delta)^{1/4}\psi\|_2.$$

The Sobolev space $H^{1/2}(\mathbb{R})$ is then defined through the norm

$$\|\psi\|_{H^{1/2}} = \|\psi\|_2 + \|(-\Delta)^{1/4}\psi\|_2.$$

Definition 2.1 *A function u is a weak L^1-energy solution to problem* (1.1) *if:*

- $u \in C([0,\infty) : L^1(\mathbb{R}))$ *and* $\log(1+u) \in L^2((0,T) : \dot{H}^{1/2}(\mathbb{R}))$ *for every* $T > 0$*;*
- *identity* (2.2) *holds for every* $\varphi \in C_0^1(\mathbb{R} \times (0,\infty))$*;*
- $u(\cdot, 0) = f$ *almost everywhere.*

For the sake of brevity, we will denote the solutions obtained below according to this definition merely as weak solutions. We remark that this is not the only way of defining a solution to problem (1.1). There are other possibilities, for instance entropy solutions [8], useful when dealing with equations involving convection terms.

As for the initial data, our concept of solution only requires in principle $f \in L^1_+(\mathbb{R})$. However, in order to prove existence we will ask f to belong to the slightly smaller $L\log L$-type space

$$\mathcal{X} = \left\{ f \geq 0 \text{ measurable} : \int_{\mathbb{R}} (1+f)\log(1+f) < \infty \right\}.$$

Notice that $L^1_+(\mathbb{R}) \cap L^p(\mathbb{R}) \subset \mathcal{X} \subset L^1_+(\mathbb{R})$ for any $p > 1$. This is to be compared with the result for the fractional porous medium equation (1.2), where in the critical case $\sigma = N(1-m)$ we have required $f \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ for some $p > 1$.

The space $\mathcal{X}$ is natural for problem (1.1). Indeed, let $\Psi(s) = (1+s)\log(1+s) - s$. A function f belongs to $\mathcal{X}$ if and only if $f \in L^1_+(\mathbb{R}^N)$ and $\int_{\mathbb{R}} \Psi(f) < \infty$. On the other hand, after an integration by parts we formally obtain

$$\int_{\mathbb{R}} \Psi(u(\cdot,t)) = \int_{\mathbb{R}} \Psi(f) - \int_0^t \int_{\mathbb{R}} \left| (-\Delta)^{1/4}(\log(1+u)) \right|^2 \leq \int_{\mathbb{R}} \Psi(f),$$

and we conclude that the space $\mathcal{X}$ is preserved by the evolution.

Notation. We will denote $L_{\mathcal{X}}(f) := \int_{\mathbb{R}} \Psi(f) = \int_{\mathbb{R}} ((1+f)\log(1+f) - f)$.

Though we will be able to prove existence of a weak solution for any $f \in \mathcal{X}$, in order to prove uniqueness we will restrict ourselves to the smaller class of *strong solutions*.

Definition 2.2 *We say that a weak solution u to problem* (1.1) *is a* strong solution *if* $\partial_t u \in L^1_{\text{loc}}((0,\infty) \times \mathbb{R})$.

If u is a strong solution, then $(-\Delta)^{1/2}\log(1+u)$ is also an L^1_{loc}-function and the equation in (1.1) is satisfied a.e.

Our first result shows that problem (1.1) is well posed in the class of strong solutions for initial data in $\mathcal{X}$.

Theorem 2.1 *For every $f \in \mathcal{X}$ there exists a unique strong solution to problem* (1.1).

Existence and uniqueness use an alternative formulation of problem (1.1) based in the Dirichlet to Neumann operator. Given a smooth bounded function $g : \mathbb{R} \mapsto \mathbb{R}$, we define its harmonic extension $v = \mathrm{E}(g)$ to the upper half-plane $\mathbb{R}^2_+$ as the unique smooth bounded solution to

$$\begin{cases} \Delta_{x,y} v = 0, & x \in \mathbb{R},\ y > 0, \\ v(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

Then, it turns out that $-\partial_y v(x,0) = (-\Delta_x)^{1/2} g(x)$, where $\Delta_{x,y}$ is the Laplacian in all (x,y)-variables and Δ_x acts only on the x-variables (in the sequel we will drop the subscripts when no confusion arises). The extension operator E can be defined by density in the space $\dot H^{1/2}(\mathbb{R})$, and it is an isometry between this space and the space $\mathcal{H}$ defined as the completion of $C_0^\infty(\overline{\mathbb{R}^2_+})$ with the norm

$$\|\psi\|_{\mathcal{H}} = \left(\int_0^\infty \int_{\mathbb{R}} |\nabla \psi|^2 \right)^{1/2}.$$

Therefore,

$$\int_{\mathbb{R}} (-\Delta)^{1/4} \phi \, (-\Delta)^{1/4} \psi = \int_0^\infty \int_{\mathbb{R}} \nabla \mathrm{E}(\phi) \cdot \nabla \mathrm{E}(\psi). \tag{2.3}$$

We also have

$$\int_0^\infty \int_{\mathbb{R}} \nabla \mathrm{E}(\phi) \cdot \nabla \mathrm{E}(\psi) = \int_0^\infty \int_{\mathbb{R}} \nabla \eta \cdot \nabla \mathrm{E}(\psi). \tag{2.4}$$

for any $\eta \in \mathcal{H}$ such that $\mathrm{Tr}(\eta) = \phi$; see [20].

Using this approach, problem (1.1) can be written in an equivalent local form. If u is a solution, then $w = \mathrm{E}(\log(1+u))$ solves

$$\begin{cases} \Delta w = 0, & (x,y) \in \mathbb{R}^2_+,\ t > 0, \\ \partial_y w - \partial_t \beta(w) = 0, & x \in \mathbb{R},\ y = 0,\ t > 0, \\ w = \log(1+f), & x \in \mathbb{R},\ y = 0,\ t = 0, \end{cases} \qquad \beta(w) = e^w - 1. \tag{2.5}$$

Conversely, if we obtain a solution w to (2.5), then $u = \beta(w)\big|_{y=0}$ is a solution to (1.1).

We next state the main properties of the solution obtained in the paper.

Theorem 2.2 *Let $f \in \mathcal{X}$. The unique strong solution u to problem* (1.1) *satisfies:*

(i) $\partial_t u \in L^2(\mathbb{R} \times (\tau, \infty))$ *for all $\tau > 0$;*

(ii) $\mathcal{X}$–L^∞ smoothing effect:

$$\|u(\cdot,t)\|_\infty \le C \max\{t^{-1} \exp(C t^{-1/2} \big(L_{\mathcal{X}}(f)\big)^{1/2}), t^{-3/4} \|f\|_1^{1/2} \big(L_{\mathcal{X}}(f)\big)^{1/4}\}; \tag{2.6}$$

(iii) $L_{\mathcal{X}}(u(\cdot,t))$ *and* $\|u(\cdot,t)\|_p$, $1 \le p \le \infty$, *are non-increasing functions of t in* $(0,\infty)$*;*

(iv) $\displaystyle\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} f(x)\,dx$ *for every* $t \geq 0$ (conservation of mass)*;*

(v) $u \in C^\infty(\mathbb{R} \times (0,\infty))$*;*

(vi) $u(x,t) > 0$ *for every* $x \in \mathbb{R}$*,* $t > 0$*.*

Plan of the paper. We will cover the existence and uniqueness theory in sections 3 and 4; we borrow results and ideas from [20]. Section 5 is devoted to obtain some basic properties of the solutions.

We then proceed with the smoothing effect, Section 6, first from $L^1 \cap L^p$, $p > 1$, to L^∞ and, then from $\mathcal{X}$ to L^2. The proof entails a number of new ideas, in particular the use of a Trudinger inequality for fractional exponents.

In Section 7 we perform a delicate regularity analysis to show that solutions are C^∞ smooth in space and time, and hence classical solutions of the equation.

We next describe in Section 8 the transformation that passes from the equation in (1.1) to the nonlocal diffusion-transport model (1.3), and the results obtained for the latter.

We finally include two appendixes. The first one is devoted to a generalization of the Nash-Trudinger type inequality used in the proof of the smoothing effect. In the second one we consider another tool used in that proof, an interesting calculus inequality.

3 Uniqueness

As mentioned in the introduction, in order to prove uniqueness we have to restrict the class of solutions under consideration. We will give two results in this direction: in the first one we restrict ourselves to weak solutions that satisfy $u \in L^2(\mathbb{R} \times (0,T))$ for all $T > 0$, and in the second to the class of strong solutions.

Theorem 3.1 *Problem* (1.1) *has at most one weak solution satisfying* $u \in L^2(\mathbb{R} \times (0,T))$ *for all* $T > 0$*.*

Proof. We adapt the classical uniqueness proof for porous medium equations due to Oleinik, Kalashnikov and Czou [18].

Let u and $\widetilde{u}$ be two weak solutions to problem (1.1). We subtract the weak formulations for u and $\widetilde{u}$ and take

$$\varphi(x,t) = \begin{cases} \displaystyle\int_t^T (\log(1+u) - \log(1+\widetilde{u}))(x,s)\,ds, & 0 \leq t \leq T, \\ 0, & t \geq T, \end{cases}$$

as a test function. Notice that, since the initial data of both solutions coincide, we do not need φ to vanish at $t = 0$. After an integration in time we get

$$\int_0^T \int_{\mathbb{R}} (u - \widetilde{u})(x,t)(\log(1+u) - \log(1+\widetilde{u}))(x,t)\,dxdt$$
$$+\frac{1}{2}\int_{\mathbb{R}} \left(\int_0^T (-\Delta)^{1/4}(\log(1+u) - \log(1+\widetilde{u}))(x,s)\,ds \right)^2 dx = 0.$$

The condition $u, \widetilde{u} \in L^2(\mathbb{R}\times(0,T))$ ensures that the first integral is well defined. Since both integrands are nonnegative, they must be identically zero. Therefore, $u = \widetilde{u}$. $\square$

Remark. In particular, for $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ there is at most a bounded weak solution.

To prove uniqueness in the class of strong solutions we use the extension technique.

Theorem 3.2 *If u and $\widetilde{u}$ are strong solutions to problem* (1.1)*, for every* $0 \le t_1 < t_2$ *we have*

$$\int_{\mathbb{R}} (u - \widetilde{u})_+(x,t_2)\,dx \le \int_{\mathbb{R}} (u - \widetilde{u})_+(x,t_1)\,dx. \tag{3.1}$$

Proof. Let p be a smooth monotone approximation to the sign function such that $0 \le p \le 1$, and let j be such that $j' = p$, $j(0) = 0$. Let $\zeta \in C_0^\infty(\mathbb{R})$ be a cut-off function, $0 \le \zeta \le 1$, $\zeta(x) = 1$ for $|x| \le 1$, $\zeta(x) = 0$ for $|x| \ge 2$, and $\zeta_R = \zeta(x/R)$.

Let $z = \log(1+u) - \log(1+\widetilde{u})$. Using (2.3) and (2.4) we get, for any $0 < t_1 < t_2$,

$$\begin{aligned}
\int_{t_1}^{t_2}\int_{\mathbb{R}} \frac{\partial(u-\widetilde{u})}{\partial t} p(z)\zeta_R &= -\int_{t_1}^{t_2}\int_{\mathbb{R}} (-\Delta)^{1/4} z\,(-\Delta)^{1/4}\,(p(z)\zeta_R) \\
&= -\int_{t_1}^{t_2}\int_0^\infty\int_{\mathbb{R}} \nabla\,\mathrm{E}(z)\cdot\nabla\,(p(E(z))E(\zeta_R)) \\
&= -\int_{t_1}^{t_2}\int_0^\infty\int_{\mathbb{R}} (p'(\mathrm{E}(z))|\nabla\,\mathrm{E}(z)|^2 E(\zeta_R) + \nabla j(\mathrm{E}(z))\cdot\nabla\,\mathrm{E}(\zeta_R)) \\
&\le -\int_{t_1}^{t_2}\int_{\mathbb{R}} (-\Delta)^{1/4} j(z)\,(-\Delta)^{1/4}\zeta_R \le \int_{t_1}^{t_2}\int_{\mathbb{R}} j(z)|(-\Delta)^{1/2}\zeta_R| \\
&\le \frac{c}{R}\int_{t_1}^{t_2}\int_{\mathbb{R}} |z| \le \frac{c(t_2-t_1)}{R}\max_{t\in[t_1,t_2]}\max\{\|u(\cdot,t)\|_1, \|\widetilde{u}(\cdot,t)\|_1\}.
\end{aligned}$$

where we have used that $|(-\Delta)^{1/2}\zeta_R(x)| = |(-\Delta)^{1/2}\zeta(x/R)|/R \le c/R$, $0 \le j(z) \le |z|$, and the fact that $\log(1+u) \le u$ for all $u \ge 0$. We end by letting $R \to \infty$ and p tend to the sign function. The case $t_1 = 0$ is obtained passing to the limit, using the L^1–continuity of $u(\cdot,t)$ at $t = 0$. $\square$

4 Existence of weak solutions

The aim of this section is to construct a weak solution for any initial data in $\mathcal{X}$. We will prove later, in section 5, that this solution, being strong, falls within the uniqueness class.

Theorem 4.1 *For every $f \in \mathcal{X}$ there exists a weak solution u to problem* (1.1). *This solution satisfies $u \geq 0$,*

$$\int_0^\infty \int_{\mathbb{R}} |(-\Delta)^{1/4} \log(1+u)|^2 \, dxdt \leq L_{\mathcal{X}}(f), \tag{4.1}$$

and, if $f \in L^\infty(\mathbb{R})$, $\|u(\cdot,t)\|_\infty \leq \|f\|_\infty$.

Proof. The construction of the solution uses several approximations. We refer to [20] for the details, where a similar calculation is made for the fractional porous medium equation (1.2).

STEP 1. We first consider initial functions $f \in L^1_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We use the formulation of the problem in the extension to $\mathbb{R}^2_+$ version (2.5). By means of the Crandall-Liggett Theorem [12] we are reduced to deal with the elliptic related problem

$$\begin{cases} \Delta w = 0, & x \in \mathbb{R},\ y > 0, \\ -\partial_y w + \beta(w) = g, & x \in \mathbb{R},\ y = 0, \end{cases} \tag{4.2}$$

with $g \in L^1_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Finally we substitute the half space $\mathbb{R}^2_+$ by a half ball $B_R^+ = \{(x,y) : |x|^2 + y^2 < R^2,\ y > 0\}$. We impose zero Dirichlet data on the "new part" of the boundary. Therefore we are led to study the problem

$$\begin{cases} \Delta w = 0 & \text{in } B_R^+, \\ w = 0 & \text{on } \partial B_R^+ \cap \{y > 0\}, \\ -\partial_y w + \beta(w) = g & \text{on } D_R := \{|x| < R,\ y = 0\}, \end{cases} \tag{4.3}$$

with $g \in L^\infty(D_R)$ given. Minimizing the functional

$$J(w) = \frac{1}{2} \int_{B_R^+} |\nabla w|^2 + \int_{D_R} (e^w - (1+g)w)$$

in the admissible set $\mathcal{A} = \{w \in H^1(B_R^+) : 0 \leq \beta(w) \leq \|g\|_\infty\}$, we obtain a unique solution $w = w_R$ to problem (4.3). Moreover, if g_1 and g_2 are two admissible data, then the corresponding weak solutions satisfy the L^1-contraction property

$$\int_{D_R} (\beta(w_1(x,0)) - \beta(w_2(x,0)))_+ \, dx \leq \int_{\mathbb{R}} (g_1(x) - g_2(x))_+ \, dx.$$

STEP 2. The passage to the limit $R \to \infty$ uses the monotonicity in R of the approximate solutions w_R. We obtain a function $w_\infty = \lim_{R\to\infty} w_R$ which is a weak solution to problem (4.2). The above contractivity property also holds in the limit. Moreover, $\|\beta(w_\infty(\cdot,0))\|_{L^\infty(\mathbb{R})} \le \|g\|_{L^\infty(\mathbb{R})}$, and $w_\infty \ge 0$, since $g \ge 0$.

Step 3. By the previous step, and using the Crandall-Liggett Theorem, we obtain the existence of a unique mild solution $\overline{w}$ to the evolution problem (2.5). To prove that $\overline{w}$ is moreover a weak solution to problem (2.5), one needs to show that it lies in the right energy space. This is done using the same technique as in [19], which yields the energy estimate

$$\int_0^T \int_0^\infty \int_{\mathbb{R}} |\nabla \overline{w}(x,y,t)|^2 \, dxdydt \le L_{\mathcal{X}}(f) \quad \text{for every } T > 0.$$

Hence the function $u = \beta(\overline{w}(\cdot,0))$ is a weak solution to problem (1.1). In addition, $\|\beta(\overline{w}(\cdot,0))\|_{L^\infty(\mathbb{R}\times(0,\infty))} \le \|f\|_{L^\infty(\mathbb{R})}$, and $\overline{w} \ge 0$. In order to obtain estimate (4.1) we recall the isometry between $\dot{H}^{1/2}(\mathbb{R})$ and $\mathcal{H}$. The Semigroup Theory also guarantees that the constructed solutions satisfy the L^1-contraction property $\|u(\cdot,t)-\widetilde{u}(\cdot,t)\|_1 \le \|f-\widetilde{f}\|_1$.

Step 4. In this last step we consider general data $f \in \mathcal{X}$. Let $\{f_k\} \subset L^1_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be a sequence of functions converging to f in $L^1(\mathbb{R})$, and let $\{u_k\}$ be the sequence of the corresponding solutions. Thanks to the L^1-contraction property we know that $u_k(\cdot,t) \to u(\cdot,t)$ in $L^1(\mathbb{R})$ for all $t > 0$ for some function u. Moreover, nonlinear Semigroup Theory guarantees that $u_k \to u$ in $C([0,\infty) : L^1(\mathbb{R}))$ [11]. On the other hand, using estimate (4.1), we have $\log(1+u_k) \in L^2((\tau,\infty) : \dot{H}^{1/2}(\mathbb{R}))$ uniformly in k. Thus the limit u is a weak solution to problem (1.1) for every $t \ge \tau$. The L^1-contraction together with the L^1-continuity allow to go down to $\tau = 0$. $\square$

5 Strong solutions and energy estimates

We still have to prove that the weak solutions that we have constructed are in fact strong. As a first step we consider the case of bounded weak solutions. The general case will follow by approximation as a consequence of the smoothing effect; see Section 6.

Proposition 5.1 *Let u be a bounded weak solution to problem* (1.1). *Then u is a strong solution and*

$$\int_t^\infty \int_{\mathbb{R}} |\partial_t u|^2 \, dxds \le ct^{-1}(1+\|u(\cdot,t)\|_\infty) L_{\mathcal{X}}(f), \qquad t > 0. \tag{5.1}$$

Proof. In order to overcome the possible lack of regularity in time, we will work with the Steklov averages of functions $g \in L^1_{\text{loc}}(\mathbb{R}\times(0,\infty))$, defined as

$$g^h(x,t) = \frac{1}{h}\int_t^{t+h} g(x,s)\, ds.$$

A similar approach is used for instance by Bénilan and Gariepy in [4] when dealing with evolution problems with standard Laplacians. The use of Steklov averages makes the process rather technical. The estimates are simpler to obtain when we assume regularity and work formally, and we invite the reader to do so. However, such regularity cannot be assumed at this stage of the theory.

Almost everywhere we have

$$\partial_t g^h(x,t) = \delta^h g(x,t) := \frac{g(x,t+h) - g(x,t)}{h}.$$

Let $h > 0$. Given any $\varphi \in C_0^\infty(\mathbb{R} \times (0,\infty))$, we may take $-\delta^{-h}\varphi$ as a test function in the weak formulation. Then, using the "integration by parts" formula $\int_0^\infty \int_{\mathbb{R}} \varphi\, \delta^h u = -\int_0^\infty \int_{\mathbb{R}} u \delta^{-h}\varphi$, we get that

$$\int_0^\infty \int_{\mathbb{R}} \varphi \delta^h u \, dxdt = -\int_0^\infty \int_{\mathbb{R}} (-\Delta)^{1/4} (\log(1+u))^h (-\Delta)^{1/4} \varphi \, dxdt.$$

Taking $\varphi = \zeta \partial_t (\log(1+u))^h$, where $\zeta = \zeta(t) \in C_0^\infty((0,\infty))$, this identity becomes

(5.2)

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} \zeta \partial_t u^h \, \partial_t (\log(1+u))^h \, dxdt &= -\frac{1}{2} \int_0^\infty \int_{\mathbb{R}} \zeta \, \partial_t \left| (-\Delta)^{1/4} (\log(1+u))^h \right|^2 \, dxdt \\ &= \frac{1}{2} \int_0^\infty \int_{\mathbb{R}} \zeta' \, \left| (-\Delta)^{1/4} (\log(1+u))^h \right|^2 \, dxdt.\end{aligned}$$

We now restrict ourselves to functions ζ which are cut-off functions for the set $[t_1, t_2]$. To be more precise, we consider $\psi \in C^\infty(\mathbb{R})$ such that $\psi' \geq 0$, $\psi(t) = 0$ for $t \leq 1/2$, $\psi(t) = 1$ for $t \geq 1$, and then define $\zeta(t) = \psi(t/t_1) - \psi(t/(2t_2))$. Then, using that $\zeta'(t) \leq t_1^{-1} \max \psi'$, together with the inequality $\delta^h u \, \delta^h \log(1+u) \geq c\,(\delta^h u)^2$, (with $c = (1 + \|u\|_\infty)^{-1}$), we get

$$c \int_{t_1}^{t_2} \int_{\mathbb{R}} (\delta^h u)^2 \, dxdt \leq \frac{1}{2t_1} \int_0^\infty \int_{\mathbb{R}} \left| (-\Delta)^{1/4} (\log(1+u))^h \right|^2 \, dxdt.$$

The energy estimate (4.1) implies that the right-hand side is bounded for h small. Therefore there is a sequence $h_n \to 0^+$ and a function $g \in L^2(\mathbb{R} \times (t,\infty))$ for all $t > 0$ such that $\delta^{h_n} u \to g$ weakly in $L^2(\mathbb{R} \times (t,\infty))$. It satisfies

$$\int_{t_1}^{t_2} \int_{\mathbb{R}} g^2 \, dxdt \leq c t_1^{-1} (1 + \|u(\cdot,t_1)\|_\infty) \int_0^\infty \int_{\mathbb{R}} \left| (-\Delta)^{1/4} \log(1+u) \right|^2 \, dxdt.$$

On the other hand,

$$\begin{aligned}-\int_0^\infty \int_{\mathbb{R}} u \partial_t \varphi \, dxdt &= -\lim_{h_n \to 0^+} \int_0^\infty \int_{\mathbb{R}} u \delta^{-h_n} \varphi \, dxdt \\ &= \lim_{h_n \to 0^+} \int_0^\infty \int_{\mathbb{R}} \delta^{h_n} u \varphi \, dxdt \\ &= \int_0^\infty \int_{\mathbb{R}} g \varphi \, dxdt,\end{aligned}$$

which means that the distributional derivative $\partial_t u$ is in fact a function that coincides with g almost everywhere. $\square$

We next prove that the L^p-norms do not increase with time. The main tool, used also later in the proof of the smoothing effect, Section 6, is the generalized Stroock-Varopoulos inequality [28], [30],

$$\int_{\mathbb{R}} A(z)(-\Delta)^{1/2}z \geq \int_{\mathbb{R}} \left|(-\Delta)^{1/4}B(z)\right|^2, \tag{5.3}$$

where $A' = (B')^2$. An easy proof using the local realization of the half-Laplacian (in a more general setting) is given in [20, Lemma 5.2].

Proposition 5.2 *Let u be a bounded weak solution to problem* (1.1). *Then, for every $0 \leq t_1 < t_2$ we have*

$$L_{\mathcal{X}}(u(\cdot,t_2)) \leq L_{\mathcal{X}}(u(\cdot,t_1)), \qquad \|u(\cdot,t_2)\|_p \leq \|u(\cdot,t_1)\|_p, \quad 1 \leq p \leq \infty.$$

Proof. The first estimate is obtained directly multiplying the equation by $\log(1+u)$, as mentioned in Section 2. The cases $p = 1$ and $p = \infty$ in the second inequality follow from the elliptic estimates in Section 4. For the rest of the cases, we put $A(z) = u^{p-1}$, $z = \log(1+u)$ in (5.3). Since $(-\Delta)^{1/2}z \in L^2(\mathbb{R})$ a.e. in t, if $p \geq 3/2$ we have $A(z) \in L^2(\mathbb{R})$. Assume this is the case. We then multiply the equation by $A(z)$ and integrate in $\mathbb{R} \times (t_1, t_2)$ to obtain

$$\frac{1}{p}\int_{\mathbb{R}} \Big(u^p(x,t_2) - u^p(x,t_1)\Big)\,dx \leq -\int_{t_1}^{t_2}\int_{\mathbb{R}} \left|(-\Delta)^{1/4}G(u)(x,t)\right|^2\,dxdt \leq 0,$$

where $G(u) = B(z) = \int_0^u \sqrt{(p-1)s^{p-2}/(1+s)}\,ds$.

For the case $1 < p < 3/2$, we approximate the function $A(z)$ by

$$A_\varepsilon(z) = \begin{cases} u^{p-1} & \text{for } u \geq \varepsilon, \\ \varepsilon^{p-2}u & \text{for } 0 \leq u < \varepsilon, \end{cases}$$

and then let ε tend to zero. $\square$

The L^1-norm is not only non-increasing; it is conserved.

Theorem 5.1 *Let u be a strong solution to problem* (1.1). *For every $t > 0$ we have*

$$\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} f(x)\,dx.$$

Proof. We take a nonnegative non-increasing cut-off function $\psi(s)$ such that $\psi(s) = 1$ for $0 \leq s \leq 1$, $\psi(s) = 0$ for $s \geq 2$, and define $\phi_R(x) = \psi(|x|/R)$. Observe that

$|(-\Delta)^{1/2}\phi_R(x)| = R^{-1}|(-\Delta)^{1/2}\psi(|x|/R)| \le c/R$. Multiplying the equation by ϕ_R and integrating by parts, we obtain, for every $t_2 > t_1 > 0$,

$$\begin{aligned}\left|\int_{\mathbb{R}}\Big(u(x,t_2)-u(x,t_1)\Big)\phi_R(x)\,dx\right| &= \left|\int_{t_1}^{t_2}\int_{\mathbb{R}}\log(1+u)(x,t)\,(-\Delta)^{1/2}\phi_R(x)\,dxdt\right|\\ &\le cR^{-1}\max_{t\in[t_1,t_2]}\|u(\cdot,t)\|_1.\end{aligned}$$

In the last step we have used that $\log(1+u)\le u$. The result is then obtained just passing to the limit $R\to\infty$. $\square$

Weak bounded solutions turn out to have an energy which is well defined for all positive times.

Proposition 5.3 *Let u be a bounded weak solution to problem* (1.1). *The energy*

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} |(-\Delta)^{1/4}\log(1+u)(x,t)|^2\,dx$$

is a continuous function in $(0,\infty)$ which does not increase with time. Moreover,

$$E(t) \le (2t)^{-1}L_{\mathcal{X}}(f) \quad \textit{for every } t>0. \tag{5.4}$$

Proof. Passing to the limit $h\to 0$ in the identity (5.2), we get

$$\int_0^\infty\int_{\mathbb{R}}\zeta\frac{|\partial_t u|^2}{1+u}\,dxdt = \frac{1}{2}\int_0^\infty\int_{\mathbb{R}}\zeta'\,\left|(-\Delta)^{1/4}\log(1+u)\right|^2\,dxdt$$

for any test function $\zeta\in C_0^\infty((0,\infty))$. This means that, as a distribution, E' coincides with the function $-\int_{\mathbb{R}}\frac{|\partial_t u|^2}{1+u}\,dx\le 0$. Since the latter belongs to $L^1(\mathbb{R})$, we conclude that $E\in W^{1,1}((0,\infty))$, and therefore that it is a continuous function. Now we have $L_{\mathcal{X}}(f)\ge 2\int_0^t E(s)\,ds \ge 2tE(t)$. $\square$

In addition to the homogeneous Sobolev space $\dot{H}^{1/2}(\mathbb{R})$, the function $\log(1+u)(\cdot,t)$ also belongs to the full space $H^{1/2}(\mathbb{R})$.

Proposition 5.4 *Let u be a bounded weak solution to problem* (1.1). *Then for every $t>0$*

$$\|\log(1+u)(\cdot,t)\|_{H^{1/2}} \le t^{-1/2}\big(L_{\mathcal{X}}(f)\big)^{1/2} + ct^{-1/4}\|f\|_1^{1/2}\big(L_{\mathcal{X}}(f)\big)^{1/4}. \tag{5.5}$$

Proof. Let $w=\log(1+u)$. We use interpolation and the Nash-Gagliardo-Nirenberg inequality (A.6) with $N=1$, $\gamma=1/2$, $q=2$, $p=1$, to get

$$\|w(\cdot,t)\|_2 \le \|w(\cdot,t)\|_3^{3/4}\|w(\cdot,t)\|_1^{1/4} \le c\|(-\Delta)^{1/4}w(\cdot,t)\|_2^{1/2}\|w(\cdot,t)\|_1^{1/2}.$$

Next we use that $\log(1+u)\le u$, the energy estimate (5.4) and the conservation of mass to conclude that

$$\|\log(1+u)(\cdot,t)\|_2 \le ct^{-1/4}\big(L_{\mathcal{X}}(f)\big)^{1/4}\|u(\cdot,t)\|_1^{1/2} = ct^{-1/4}\big(L_{\mathcal{X}}(f)\big)^{1/4}\|f\|_1^{1/2}. \tag{5.6}$$

$\square$

To end this section, we improve the regularity of u and $\log(1+u)$, giving an L^2-control of their gradients.

Corollary 5.1 *Let u be a bounded weak solution to problem* (1.1). *Then u and $\log(1+u)$ belong to $L^2_{\rm loc}((0,\infty):H^1(\mathbb{R}))$, and*

$$\int_t^\infty \int_{\mathbb{R}} |\partial_x \log(1+u)(x,s)|^2\,dxds \le ct^{-1}(1+\|u(\cdot,t)\|_\infty)L_{\mathcal{X}}(f),$$
$$\int_t^\infty \int_{\mathbb{R}} |\partial_x u(x,s)|^2\,dxds \le ct^{-1}(1+\|u(\cdot,t)\|_\infty)^3 L_{\mathcal{X}}(f).$$

Proof. It is clear from (5.6) that $\log(1+u)\in L^2((0,T):L^2(\mathbb{R}))$ for every $T>0$. To estimate the gradient we just use (5.1) and the equation. Actually,

$$\begin{aligned}\int_t^\infty \int_{\mathbb{R}} |\partial_x \log(1+u)(x,s)|^2\,dxds &= \int_t^\infty \int_{\mathbb{R}} |(-\Delta)^{1/2}\log(1+u)(x,s)|^2\,dxds\\ &= \int_t^\infty \int_{\mathbb{R}} |\partial_t u(x,s)|^2\,dxds \le ct^{-1}(1+\|u(\cdot,t)\|_\infty)L_{\mathcal{X}}(f).\end{aligned}$$

As to u, we just observe that $\partial_x u=(1+u)\partial_x\log(1+u)$. $\square$

6 Smoothing effect

In Section 4 we have constructed a weak solution of problem (1.1) for general initial data $f\in\mathcal{X}$ by approximation with initial data in $L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$. Our next aim is to prove that this solution becomes immediately bounded; in particular it is strong. Boundedness will follow from an estimate for bounded weak solutions, formula (2.6), which *does not depend* on the L^∞ norm of the datum, but only on $L_{\mathcal{X}}(f)$ (and time).

The result will be obtained by combining $L^2\to L^\infty$ and $\mathcal{X}\to L^2$ smoothing effects. The $L^2\to L^\infty$ result is in fact a particular instance of a more general $L^p\to L^\infty$ result, valid for all $p>1$.

Theorem 6.1 *Let u be a bounded weak solution, and let $p>1$. There is a constant $C>0$ that depends only on p such that*

$$\|u(\cdot,t)\|_\infty \le C\,\max\{t^{-1/(p-1)}\|f\|_p^{p/(p-1)}, t^{-1/p}\|f\|_p\}. \tag{6.1}$$

We recall that the corresponding formula for the fractional PME with $m>0$ reads, in the case $N=\sigma=1$,

$$\|u(\cdot,t)\|_\infty \le C\,t^{-1/(m+p-1)}\|f\|_p^{p/(m+p-1)}, \tag{6.2}$$

for every $p \geq 1$, cf. [20]. Observe that when $m = 0$ these exponents make sense for $p > 1$ but not for $p = 1$. It is also worth noticing that formula (6.1) can be obtained by formally putting in (6.2) $m = 0$ for u large and $m = 1$ for u small.

Proof. The proof follows the same Moser iterative technique used in [20], but it is a little more involved. Let $t > 0$ be fixed, and consider the sequence of times $t_k = (1 - 2^{-k})t$, $p_k = 2^k p$. We multiply the equation (recall that it is satisfied a.e. since u is a strong solution) by the test function

$$\phi = \frac{u^{p_k-1}}{p_k - 1} + \frac{u^{p_k}}{p_k}$$

and integrate in $\mathbb{R} \times (t_k, t_{k+1})$ (for $p_0 = p \in (1, 3/2)$ we need an extra approximation argument, as in Proposition 5.2, to justify the computation). Using now the Stroock-Varopoulos inequality (5.3), we get

$$\begin{aligned} \frac{1}{p_k(p_k-1)}\|u(\cdot,t_k)\|_{p_k}^{p_k} &+ \frac{1}{p_k(p_k+1)}\|u(\cdot,t_k)\|_{p_k+1}^{p_k+1} \\ &\geq \frac{4}{p_k^2}\int_{t_k}^{t_{k+1}}\int_{\mathbb{R}} |(-\Delta)^{1/4}u^{p_k/2}(x,\tau)|^2\,dxd\tau. \end{aligned}$$

Multiplying and dividing by $\|u(\cdot,\tau)\|_r^r$, for some $r > 1$, $r \geq p_k/2$, using that the L^r norms do not increase in time, and applying the Nash-Gagliardo-Nirenberg type inequality (A.6) with $N = 1$, $\gamma = 1/2$, $q = 2$, we get

$$\|u(\cdot,t_{k+1})\|_{p_k+r}^{p_k+r} \leq c2^k t^{-1}\|u(\cdot,t_k)\|_r^r\Big(\|u(\cdot,t_k)\|_{p_k}^{p_k} + \|u(\cdot,t_k)\|_{p_k+1}^{p_k+1}\Big). \tag{6.3}$$

Let us denote $U_k = \max\{\|u(\cdot,t_k)\|_{p_k}, \|u(\cdot,t_k)\|_{p_k+1}^{(p_k+1)/p_k}\}$. Taking $r = p_k$ and $r = p_k + 1$ in (6.3) we get that both $\|u(\cdot,t_{k+1})\|_{p_{k+1}}^{p_{k+1}}$ and $\|u(\cdot,t_{k+1})\|_{p_{k+1}+1}^{p_{k+1}+1}$ are smaller than $c2^k t^{-1}U_k^{p_{k+1}}$, from where we obtain

$$U_{k+1} \leq (c2^k t^{-1})^{1/(p_{k+1})}U_k = (c2^{k/p}t^{-1/p})^{1/2^{k+1}}U_k.$$

This recursive relation yields

$$\|u(\cdot,t)\|_\infty = \lim_{k\to\infty} U_k \leq ct^{-1/p}U_0 = ct^{-1/p}\max\{\|f\|_p\,, \|f\|_{p+1}^{(p+1)/p}\}.$$

The final step is to get rid of the L^{p+1}-norm. Using Hölder's inequality and the decay of the L^p-norms we get

$$\|u(\cdot,t)\|_\infty \leq c(t/2)^{-1/p}\|f\|_p \max\{1, \|u(\cdot,t/2)\|_\infty^{1/p}\}.$$

If $\|u(\cdot,t/2)\|_\infty \leq 1$, then $\|u(\cdot,t)\|_\infty \leq c2^{1/p}t^{-1/p}\|f\|_p$ and we are done. If, on the contrary, $\|u(\cdot,t/2)\|_\infty \geq 1$, we have

$$\|u(\cdot,t)\|_\infty \leq c(t/2)^{-1/p}\|f\|_p\|u(\cdot,t/2)\|_\infty^{1/p}.$$

Since in this case, by the maximum principle, we have $\|u(\cdot,\tau)\|_\infty \geq 1$ for every $0<\tau<t/2$, we may iterate this estimate to get

$$\|u(\cdot,t)\|_\infty \leq ct^{-1/(p-1)}\|f\|_p^{p/(p-1)}.$$

□

The above method does not allow to go down to $p=1$. This drawback was already present in the PME case (both local and nonlocal, see [32] and [20]), where the limit exponent was $p=\max\{1,\,(1-m)N/\sigma\}$. In the case $N=\sigma=1$, and putting $m=0$, we get that the limit exponent should be $p=1$, but it is not clear if solutions will become bounded when the initial datum only belongs to $L^1(\mathbb{R})$. Nevertheless, we may consider initial values in the slightly smaller space $\mathcal{X}$. This is our next goal.

Theorem 6.2 *Let u be a bounded weak solution. There is a constant $C>0$ such that*

$$\int_{\mathbb{R}} u^2(x,t)\,dx \leq \exp\left\{C\left(t^{-1/2}\big(L_{\mathcal{X}}(f)\big)^{1/2}+t^{-1/4}\|f\|_1^{1/2}\big(L_{\mathcal{X}}(f)\big)^{1/4}\right)\right\}-1. \tag{6.4}$$

Proof. Fix any time $t>0$ and let $w=\log(1+u(\cdot,t))$. We know from Proposition 5.4 that $w\in H^{1/2}(\mathbb{R})$. Hence, using the Trudinger type inequality (A.4), with $N=1$ and $\gamma=1/2$, we obtain

$$\int_{\mathbb{R}}\left(e^{w^2/c\|w\|^2_{H^{1/2}}}-1\right)\leq 1.$$

We now apply the calculus inequality $(e^w-1)^2\leq(e^k-1)(e^{w^2/k}-1)$ (see Lemma B.1 below for the proof), to get

$$\int_{\mathbb{R}} u^2(x,t)\,dx \leq \exp(c\|w\|^2_{H^{1/2}})-1.$$

We conclude using the energy estimate (5.5). □

To obtain the $\mathcal{X}-L^\infty$ smoothing effect we just have to combine Theorems 6.1 and 6.2.

Proof of Theorem 2.2-(ii). We first consider the case of initial data which are moreover bounded. The general case is dealt with by approximation.

Using the L^p-L^∞ estimate (6.1) with $p=2$, and the $\mathcal{X}$-L^2 estimate (6.4) we get, first for t small,

$$\|u(\cdot,t)\|_\infty \leq Ct^{-1}\|u(\cdot,t/2)\|_2^2 \leq Ct^{-1}\exp(Ct^{-1/2}L_{\mathcal{X}}(f)^{1/2}),$$

and then for t large

$$\|u(\cdot,t)\|_\infty \leq Ct^{-1/2}\|u(\cdot,t/2)\|_2 \leq Ct^{-3/4}\|f\|_1^{1/2}L_{\mathcal{X}}(f)^{1/4}.$$

Combining both estimates we obtain (2.6). □

7 Regularity and positivity

The solution that we have constructed in the previous sections is C^∞ for all positive times, and hence classical. This is the content of the present section.

7.1 $C^{1,\alpha}$ regularity

The first and more difficult step is to prove that the solution u is $C^{1,\alpha}$ for all $\alpha \in (0,1)$. Actually, given $\tau > 0$, u is uniformly $C^{1,\alpha}$ in $Q_\tau = \mathbb{R} \times (\tau, \infty)$, denoted $u \in C^{1,\alpha}_{\rm u}(Q_\tau)$ for short.

Theorem 7.1 *Let $f \in \mathcal{X}$. The strong solution to problem* (1.1) *satisfies $u \in C^{1,\alpha}_{\rm u}(Q_\tau)$ for every $0 < \alpha < 1$ and $\tau > 0$.*

Proof. Step 1: *$u \in C^{\alpha}_{\rm u}(Q_\tau)$ for some $\alpha \in (0,1)$ and every $\tau > 0$.*

Once we know that u is bounded in the time interval $t \geq \tau > 0$, the result follows from the regularity results for problem (2.5) from Athanasopoulos-Caffarelli [3], since the nonlinearity $\beta(u)$ satisfies the non-degeneracy condition required in that paper.

Step 2: *$u \in C^{\alpha}_{\rm u}(Q_\tau)$ for every $0 < \alpha < 1$ and every $\tau > 0$.*

To prove this, we will show that Hölder regularity can be "doubled", following ideas from Caffarelli and Vasseur [6]; i.e., if $u \in C^{\alpha}_{\rm u}(Q_\tau)$ for some $\alpha \in (0,1/2)$, then $u \in C^{2\alpha}_{\rm u}(Q_\tau)$. The claimed regularity is then obtained repeating the argument a finite number of times.

Let $(x_0,t_0) \in Q_\tau$ be fixed, and denote $u_0 = u(x_0,t_0)$. We write the equation in (1.1) as a fractional linear heat equation with a (nonlinear) source term,

$$\partial_t u + \mu(-\Delta)^{1/2}u = -(-\Delta)^{1/2}(\log(1+u) - \mu u). \tag{7.1}$$

If we take $\mu = 1/(1+u_0)$, the right-hand side of equation (7.1) can be written as $-(-\Delta)^{1/2}F(u)$, where $F(u) = \log(\mu(1+u)) - \mu(u-u_0)$ satisfies $F(u_0) = F'(u_0) = 0$. After a time shift, we may assume that u is uniformly C^α and bounded down to $t = 0$. Recall now that the fundamental solution to the fractional heat equation $\partial_t u + (-\Delta)^{1/2}u = 0$ is the Poisson kernel

$$P(x,t) = \frac{1}{\pi}\frac{t}{x^2+t^2}.$$

Taking a smooth approximation of $P(x,\mu t)$ as a test function in the distributional version of (7.1), and passing to the limit in the approximation we get that the solution u can be represented in the (mild solution) form

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}} P(x-x_1,\mu t)f(x_1)\,dx_1 \\ &\quad - \int_0^t \int_{\mathbb{R}} (-\Delta)^{1/2}P(x-x_1,\mu(t-t_1))F(u(x_1,t_1))\,dx_1 dt_1. \end{aligned} \tag{7.2}$$

The first term in the right-hand side of (7.2) is regular, so we concentrate on the second one.

We will use the notation $y = (x,t)$ for the space-time variable, and also $\overline{y} = (x, \mu t)$ to accommodate the distortion in time created by the factor μ. We are thus led to study the regularity for the function

$$g(y) = \int_{\mathbb{R}^2_+} A(\overline{y} - \overline{y}_1)\chi_{\{t_1<t\}} F(u(y_1))\, dy_1, \tag{7.3}$$

where

$$A(y) = A(x,t) \equiv (-\Delta)^{1/2} P(x,t) = \frac{1}{\pi}\frac{x^2-t^2}{(x^2+t^2)^2}.$$

Let us see first that the function g is well defined. To this aim we decompose $\mathbb{R}^2_+$ as $E_\rho \cup E_\rho^c$, where E_ρ is the ellipse

$$E = \{y_1 \in \mathbb{R}^2_+ \,:\, |\overline{y}_1 - \overline{y}| < \rho\},$$

with ρ small. Observing that

$$\int_{E_\rho} A(\overline{y} - \overline{y}_1)\chi_{\{t_1<t\}}\, dy_1 = \frac{1}{\pi\mu}\int_{\{x^2+t^2<\rho^2,\, t<0\}} \frac{x^2-t^2}{(x^2+t^2)^2}\, dxdt = 0,$$

we may write

$$\begin{aligned} \left|\int_{E_\rho} A(\overline{y} - \overline{y}_1)\chi_{\{t_1<t\}} F(u(y_1))\, dy_1\right| &\leq \int_{E_\rho} |A(\overline{y} - \overline{y}_1)|\, |F(u(y_1)) - F(u(y))|\, dy_1 \\ &\leq c\int_{E_\rho} \frac{dy_1}{|y-y_1|^{2-\beta}} \leq c \end{aligned}$$

for some $\beta > 0$, since both u and F are Hölder continuous functions. On the other hand, using that $F(u)$ is bounded we have

$$\left|\int_{E_\rho^c} A(\overline{y} - \overline{y}_1)\chi_{\{t_1<t\}} F(u(y_1))\, dy_1\right| \leq c\int_0^t \int_{|x_1-x|>1/(2\mu)} \frac{dx_1 dt_1}{|x-x_1|^2} \leq c.$$

Now we will see that $g(y)$ has the same regularity as $F(u(y))$. The key point is that if u is C^α at y_0, then $F(u(y))$ is $C^{2\alpha}$ at y_0. Indeed, since $F(u_0) = F'(u_0) = 0$, and $|F''(u)| \leq c$ (recall that $u \geq 0$), we have

$$|F(u(y))| \leq c|u(y) - u_0|^2 \leq c|y - y_0|^{2\alpha}$$

for every $y \in \mathbb{R}^2_+$. Moreover, the constants are independent of the point y_0. We observe also that $|A(\overline{y})| \leq c|A(y)|$, where $c = c(\mu)$. Since u is bounded and nonnegative, the constant $c(\mu)$ can be taken independent of μ.

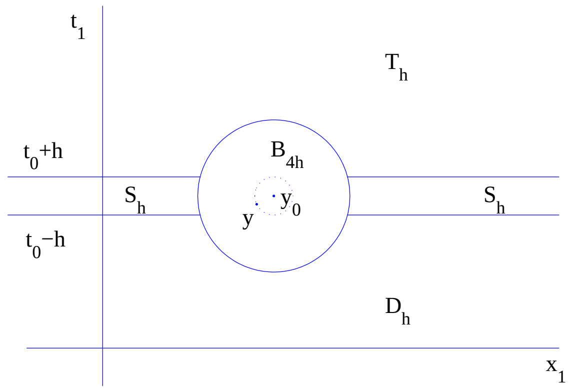

Figure 1: Integration regions.

Let $y \in \mathbb{R}^2_+$ be any point with $|y - y_0| = h$. We have to prove that the difference

$$(7.4) \qquad g(y_0) - g(y) = \int_{\mathbb{R}^2_+} \Big(A(\overline{y}_0 - \overline{y}_1)\chi_{\{t_1<t_0\}} - A(\overline{y} - \overline{y}_1)\chi_{\{t_1<t\}} \Big) F(u(y_1))\, dy_1$$

is $O(h^{2\alpha})$ for h small. In order to estimate the integral in (7.4) we decompose $\mathbb{R}^2_+$ into four regions, depending on the sizes of $|x_1 - x_0|$ and $t_1 - t_0$, see Figure 1.

(i) *The small ball* $B_{4h} = \{|y_1 - y_0| < 4h\} \subset \mathbb{R}^2_+$. The difficulty in this region is the non-integrable singularity of $A(\overline{y})$ at $\overline{y} = 0$. Integrability will be gained thanks to the regularity of $F(u)$. We have,

$$\int_{B_{4h}} |A(\overline{y}_0 - \overline{y}_1)|\, |F(u(y_1))|\, dy_1 \leq c \int_{B_{4h}} \frac{dy_1}{|y_1 - y_0|^{2-2\alpha}} \leq ch^{2\alpha}.$$

In order to estimate $\int_{B_{4h}} A(\overline{y} - \overline{y}_1) F(u(y_1))\, dy_1$, we consider as before the ellipse E_{ch}, where $c = c(\mu)$ is chosen to have $E_{ch} \subset B_{2h}$, see Figure 2.

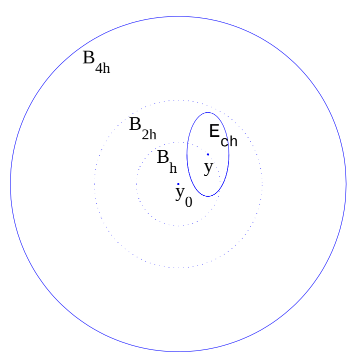

Figure 2: Integration subregions in B_{4h}.

We get

$$\int_{B_{4h}} A(\overline{y}-\overline{y}_1)\chi_{\{t_1<t\}}F(u(y_1))\,dy_1 = \underbrace{\int_{B_{4h}} A(\overline{y}-\overline{y}_1)\chi_{\{t_1<t\}}\Big(F(u(y_1))-F(u(y))\Big)\,dy_1}_{I_1} \underbrace{+\,F(u(y))\int_{B_{4h}-E_{ch}} A(\overline{y}-\overline{y}_1)\chi_{\{t_1<t\}}\,dy_1}_{I_2},$$

since the integral over the ellipse is again zero by symmetry. To estimate I_1 we use the Mean Value Theorem applied to the function F to see that

$$|F(u(y))-F(u(y_1))| = |F'(\theta)|\,|u(y)-u(y_1)| \le c\max\{|u(y)-u_0|,|u(y_1)-u_0|\}|y-y_1|^{\alpha},$$

where θ is some value between $u(y)$ and $u(y_1)$. Therefore,

$$|I_1| \le c\int_{B_{4h}} \frac{1}{|y-y_1|^2}|y-y_1|^{\alpha}(|y_1-y_0|^{\alpha}+|y-y_0|^{\alpha})\,dy_1 \le ch^{2\alpha}.$$

As to I_2, since we are far from the singularity of A,

$$|I_2| \le ch^{2\alpha}\int_{B_{4h}-E_{ch}} \frac{dy_1}{h^2} \le ch^{2\alpha}.$$

(ii) *The narrow strip* $S_h = \{|y_1-y_0|>4h,\ |t_1-t_0|<h\}$. In this region we have $|y_0-y_1| \le \frac{4}{3}|y-y_1|$ and $|x_1-x_0|>3h$. Therefore,

$$\begin{aligned}
&\int_{S_h} |A(\overline{y}_0-\overline{y}_1)\chi_{\{t_1<t_0\}} - A(\overline{y}-\overline{y}_1)\chi_{\{t_1<t\}}||F(u(y_1))|\,dy_1\\
&\quad\le \int_{S_h}\Big(|A(\overline{y}_0-\overline{y}_1)|+|A(\overline{y}-\overline{y}_1)|\Big)|F(u(y_1))|\,dy_1 \le \int_{S_h}\frac{dy_1}{|y_0-y_1|^{2-2\alpha}}\\
&\quad\le c\int_{t_0-h}^{t_0+h}\int_{|x_1-x_0|>3h}\frac{dx_1dt_1}{|x_0-x_1|^{2-2\alpha}} \le ch^{2\alpha}.
\end{aligned}$$

(iii) *The complement of the ball* B_{4h} *for large times,* $T_h = \{|y_1-y_0|>4h,\ t_1>t_0+h\}$. The integral in this region is 0, since here we have

$$A(\overline{y}_0-\overline{y}_1)\chi_{\{t_1<t_0\}} = A(\overline{y}-\overline{y}_1)\chi_{\{t_1<t\}} = 0.$$

(iv) *The complement of the ball* B_{4h} *for small times,* $D_h = \{|y_1-y_0|>4h,\ t_1<t_0-h\}$. The required estimate is obtained here using the fact that we are integrating a difference of A's, so there will be some cancelation. Indeed, by the Mean Value Theorem,

$$|A(\overline{y}_0-\overline{y}_1)-A(\overline{y}-\overline{y}_1)| \le |\overline{y}_0-\overline{y}|\max\{|\partial_x A(\xi)|,|\partial_t A(\xi)|\} \le ch/|\xi|^3,$$

where $\xi = s(\overline{y}_0 - \overline{y}_1) + (1-s)(\overline{y} - \overline{y}_1)$ for some $s \in (0,1)$. On the other hand, since we are in D_h, $|\overline{y}_0 - \overline{y}_1| \le \mu^{1/2}|y_0 - y_1| \le \frac{4\mu(1-s)}{3}|\xi| \le c|\xi|$, and we conclude that

$$|A(\overline{y}_0 - \overline{y}_1)\chi_{\{t_1<t_0\}} - A(\overline{y} - \overline{y}_1)\chi_{\{t_1<t\}}| \le \frac{ch}{|y_0 - y_1|^3}.$$

Therefore, assuming that $\alpha < 1/2$,

$$\int_{D_h} |A(\overline{y}_0 - \overline{y}_1)\chi_{\{t_1<t_0\}} - A(\overline{y} - \overline{y}_1)\chi_{\{t_1<t\}}|\,|F(u(y_1))|\,dy_1 \\ \le ch \int_{D_h} \frac{dy_1}{|y_0 - y_1|^{3-2\alpha}} \le ch^{2\alpha}.$$

Step 3: *$u \in C^{1,\alpha}_{\mathrm{u}}(Q_\tau)$ for every $0 < \alpha < 1$ and every $\tau > 0$.*

We may assume, after a time shift, that $\tau = 0$. Let $z = y - y_0$. The result will follow from an estimate of the quantity

$$g(y_0 + z) - 2g(y_0) + g(y_0 - z) = \int_{\mathbb{R}^2_+} \mathcal{A}(y_0, y, y_1) F(u(y_1))\,dy_1,$$

where

$$\mathcal{A}(y_0, y, y_1) = A(\overline{y} - \overline{y}_1)\chi_{\{t_1<t\}} - 2A(\overline{y}_0 - \overline{y}_1)\chi_{\{t_1<t_0\}} + A(2\overline{y}_0 - \overline{y} - \overline{y}_1)\chi_{\{t_1<2t_0-t\}}.$$

As in the previous step, we consider separately the contributions to the integral of the four regions shown in Figure 1. The contribution of the ball B_{4h} is decomposed as the sum $J_1 - 2J_2 + J_3$, where

$$\begin{aligned} J_1 &= \int_{B_{4h}} A(\overline{y} - \overline{y}_1)\chi_{\{t_1<t\}} F(u(y_1))\,dy_1, \\ J_2 &= \int_{B_{4h}} A(\overline{y}_0 - \overline{y}_1)\chi_{\{t_1<t_0\}} F(u(y_1))\,dy_1, \\ J_3 &= \int_{B_{4h}} A(\overline{\eta} - \overline{y}_1)\chi_{\{t_1<2t_0-t\}} F(u(y_1))\,dy_1, \qquad \overline{\eta} = 2\overline{y}_0 - \overline{y}. \end{aligned}$$

The integrals J_1 and J_2 were already estimated in Step 2. Since $|\overline{\eta} - \overline{y}_0| = |\overline{y} - \overline{y}_0|$, the integral J_3 is estimated just in the same way as J_1.

The contribution of S_h is estimated in the same way as in Step 2, just using a rough estimate of the A's. The contribution of T_h is obviously 0.

As for D_h, in this region we have, using Taylor's formula,

$$|\mathcal{A}(y_0, y, y_1)| = |A(\overline{y} - \overline{y}_1) - 2A(\overline{y}_0 - \overline{y}_1) + A(2\overline{y}_0 - \overline{y} - \overline{y}_1)| \le \frac{ch^2}{|y_0 - y_1|^4}.$$

Since $u \in C^\alpha_{\mathrm{u}}(\mathbb{R} \times (0,\infty))$ for all $\alpha \in (0,1)$, we obtain

$$\int_{D_h} |\mathcal{A}(y_0, y, y_1)|\,|F(u(y_1))|\,dy_1 \le ch^2 \int_{D_h} \frac{dy_1}{|y_0 - y_1|^{4-2\alpha}} \le ch^{2\alpha}.$$

In summary we get

$$g(y_0+z)-2g(y_0)+g(y_0-z)=O(|z|^{2\alpha})$$

for every $\alpha\in(0,1)$, $|z|<|y_0|$ (uniformly in $y_0\in\mathbb{R}\times(0,\infty)$). This estimate, together with the fact that g is bounded, allows to prove that $(-\Delta)^{\sigma/2}g(y)$ is bounded in $Q_{\tau'}$ for every $\sigma\in(0,2)$ and $\tau'>0$. Indeed, if $\sigma\in(0,2\alpha)$ and $y\in Q_{\tau'}$, we have

$$\begin{aligned}|(-\Delta)^{\sigma/2}g(y)| &= \left|c_\sigma\int_{\mathbb{R}^2}\frac{g(y+z)-2g(y)+g(y-z)}{|z|^{2+\sigma}}dz\right|\\ &\le c\int_{\{|z|<\tau'\}}\frac{|z|^{2\alpha}}{|z|^{2+\sigma}}dz+c\int_{\{|z|>\tau'\}}\frac{dz}{|z|^{2+\sigma}}\le c.\end{aligned}$$

Then, arguing in the same way as in [22, Proposition 2.9] (where the boundedness of the fractional Laplacian is assumed in the whole $\mathbb{R}^2$, not only in a half-plane), if we take $\alpha\in(1/2,1)$ and $\sigma\in(1,2\alpha)$, we obtain $g\in C^{1,\beta}(Q_{\tau'})$ for every $\beta\in(0,\sigma-1)$, with uniform norm. We conclude that $g\in C^{1,\alpha}_{\mathrm{u}}(Q_\tau)$ for every $\alpha\in(0,1)$, $\tau>0$. $\square$

7.2 C^∞ regularity

Further regularity will now be a consequence of a result for linear equations with smooth coefficients which has independent interest.

Theorem 7.2 *Let v be a bounded weak solution to $\partial_t v+(-\Delta)^{1/2}(av+b)=0$, where the coefficients satisfy $a,\, b\in C^{1,\alpha}_{\mathrm{u}}(\mathbb{R}\times(0,\infty))\cap L^\infty(\mathbb{R}\times(0,\infty))$, $a(x,t)>0$. If $v\in C^{\alpha}_{\mathrm{u}}(\mathbb{R}\times(0,\infty))$ then $v\in C^{1,\alpha}_{\mathrm{u}}(\mathbb{R}\times(\tau,\infty))$ for every $\tau>0$.*

Proof. Let $(x_0,t_0)\in\mathbb{R}^2_+$ be fixed and denote $v_0=v(x_0,t_0)$, $a_0=a(x_0,t_0)$. Then v is a distributional solution to the inhomogeneous fractional heat equation

$$\partial_t v+a_0(-\Delta)^{1/2}v=(-\Delta)^{1/2}F_1+(-\Delta)^{1/2}F_2,$$

where

$$F_1=-(a-a_0)(v-v_0),\qquad F_2=-b-v_0a.$$

Reasoning like in the proof of Theorem 7.1, we are reduced to check that

$$f_i(x,t)=\int_0^t\int_{\mathbb{R}}(-\Delta)^{1/2}P(x-x_1,a_0(t-t_1))F_i(x_1,t_1)\,dx_1dt_1,\qquad i=1,2$$

are $C^{1,\alpha}$ functions, with uniform norm. It is clear that f_2 inherits the regularity of F_2; as to f_1, we use the fact that the product $(a-a_0)(v-v_0)$ is $C^{2\alpha}$ (or $C^{1,2\alpha-1}$ if $\alpha>1/2$) whenever v is C^α. $\square$

Corollary 7.1 *The strong solution to problem* (1.1) *belongs to $C^\infty_{\mathrm{u}}(Q_\tau)$ for every $\tau>0$.*

Proof. The proof proceeds by induction. We know that $u \in C^{1,\alpha}_{\mathrm{u}}(Q_\tau)$, $\alpha \in (0,1)$, $\tau > 0$. Assume that we have already shown that $u \in C^{k,\alpha}_{\mathrm{u}}(Q_\tau)$ for some $k \geq 1$. Then, $v_k = \partial_t^\beta \partial_x^\gamma u$, $\beta + \gamma = k$, satisfies an equation of the form $\partial_t v_k + (-\Delta)^{1/2}(a_k v_k + b_k) = 0$. Let us check that the coefficients satisfy the hypotheses of Theorem 7.2. On one hand, for all $k \geq 1$, $a_k = 1/(1+u)$ is $C^{k,\alpha}$, hence $C^{1,\alpha}$. It is also bounded, since u is nonnegative. On the other hand, as $u \in C^{k,\alpha}_{\mathrm{u}}(Q_\tau) \cap L^\infty(Q_\tau)$, we obtain $v_k \in C^{\alpha}_{\mathrm{u}}(Q_\tau) \cap L^\infty(Q_\tau)$. What is left is to verify that b_k has the required regularity. In the case $k = 1$ we have $b_1 = 0$, and there is nothing to prove. When $k = 2$ we have three cases,

$$b_2 = \frac{(\partial_t u)^2}{(1+u)^2}, \quad \text{or} \quad b_2 = \frac{(\partial_x u)^2}{(1+u)^2}, \quad \text{or} \quad b_2 = \frac{\partial_t u \partial_x u}{(1+u)^2}.$$

Since $u \in C^{2,\alpha}_{\mathrm{u}}(Q_\tau) \cap L^\infty(Q_\tau)$, we have clearly $b_2 \in C^{1,\alpha}_{\mathrm{u}}(Q_\tau) \cap L^\infty(Q_\tau)$. Applying Theorem 7.2, we obtain $v_2 \in C^{1,\alpha}_{\mathrm{u}}(Q_{\tau'})$, $\tau' > \tau$. Hence $u \in C^{3,\alpha}_{\mathrm{u}}(Q_{\tau'})$.

The same reasoning works for every $k \in \mathbb{N}$. Indeed, the recursion formula for the coefficients b_k has the form

$$b_k = \partial_i b_{k-1} + v_{k-1}\, \partial_i a\,,$$

where $i = x$ or $i = t$. We observe that b_k is a polynomial in $\partial_t^{\beta'} \partial_x^{\gamma'} u$, $0 \leq \beta' \leq \beta$, $0 \leq \gamma' \leq \gamma$, $1 \leq \beta' + \gamma' \leq k - 1$, with coefficients involving the powers $(1+u)^{-m}$, $0 < m \leq k$. By the induction hypothesis, $b_k \in C^{1,\alpha}_{\mathrm{u}}(Q_\tau)$. As in the step $k = 2$ we conclude $u \in C^{k+1,\alpha}_{\mathrm{u}}(Q_{\tau'})$. □

7.3 Positivity

Once the solution is regular and the equation is satisfied in the classical sense, we can use the Riesz representation (2.1) for the fractional Laplacian. Hence, at any point (x_0, t_0) at which we have $u(x_0, t_0) = 0$, we obtain

$$\partial_t u(x_0, t_0) = \frac{1}{\pi}\ \mathrm{P.V.} \int_{\mathbb{R}} \frac{\log(1 + (u(s, t_0)))}{|x_0 - s|^2}\, ds.$$

Since u is nonnegative, the right-hand side is nonnegative. Moreover, thanks to the conservation of mass, we know that the solution u is nontrivial if $f \not\equiv 0$. Hence $\partial_t u(x_0, t_0)$ is strictly positive. We have thus proved the following positivity result.

Theorem 7.3 *If $f \not\equiv 0$, the solution to problem* (1.1) *is positive for all $x \in \mathbb{R}$ and $t > 0$.*

8 A nonlocal transport equation

We first recall that the half-Laplacian $(-\Delta)^{1/2}$ can be written in terms of the Hilbert transform as $(-\Delta)^{1/2} = H\partial_x = \partial_x H$. The latter equality holds provided that the operators are acting on a function belonging to some $W^{1,p}(\mathbb{R})$ space, $p > 1$.

We now consider the change of variables $(x,t,u)\mapsto(y,\tau,v)$ given by the Bäcklund type transform

$$y=\int_0^x (1+u(s,t))\,ds-c(t),\quad \tau=t,\qquad v(y,\tau)=\log(1+u(x,t))$$

with $c'(t)=H(\log(1+u))(0,t)$. We denote $(y,\tau)=J(x,t)$. Notice that the Jacobian of the transformation J is $\frac{\partial(y,\tau)}{\partial(x,t)}=1+u\neq 0$, since $u\geq 0$. Then we may write the inverse

$$x=\int_0^y e^{-v(\sigma,\tau)}\,d\sigma-\overline{c}(\tau),$$

with $\overline{c}\,'(\tau)=-H(\log(1+u))(0,t)/(1+u(0,t))$.

We have

$$\partial_x y=1+u,\quad \partial_t y=-H(\log(1+u))=-\widetilde{H}(v),$$

where $\widetilde{H}(v)=H(v\circ J)\circ J^{-1}$ is the conjugate of the Hilbert transform H by the transformation J. Specifically,

$$\begin{aligned}\widetilde{H}(v(y,\tau))=H(\log(1+u(x,t)))\ &=\frac{1}{\pi}\mathrm{P.V.}\int_{\mathbb{R}}\frac{\log(1+u(x',t))}{x-x'}\,dx'\\ &=\frac{1}{\pi}\mathrm{P.V.}\int_{\mathbb{R}}\frac{v(y',\tau)}{\int_{y'}^{y}e^{v(y',\tau)-v(\sigma,\tau)}\,d\sigma}\,dy'\,.\end{aligned}$$

With all this, equation (1.1) becomes

$$\partial_\tau v-\widetilde{H}(v)\,\partial_y v+\partial_y\widetilde{H}(v)=0, \tag{8.1}$$

where $y\in\mathbb{R}$, $\tau>0$. Since we assume $u\geq 0$ we get $v\geq 0$.

The L^1 norms of these two variables are related by

$$\int_{\mathbb{R}} u(x,t)\,dx=\int_{\mathbb{R}}(1-e^{-v(y,\tau)})\,dy\,,$$

and

$$\int_{\mathbb{R}} v(y,\tau)\,dy=\int_{\mathbb{R}}(1+u(x,t))\log(1+u(x,t))\,dx\,.$$

In particular, $v_0\in L^1(\mathbb{R})$ if and only if $u_0\in\mathcal{X}$. On the other hand,

$$\partial_y v=\partial_x u,\qquad \partial_\tau v=\frac{1}{1+u}\partial_t u+H(\log(1+u))\partial_x u.$$

This allows to obtain regularity results for v from smoothness results for u. Finally, we have

$$\begin{aligned}\int_{\mathbb{R}}|(-\Delta)^{1/4}v(y,\tau)|^2\,dy&=\int_{\mathbb{R}}|(-\Delta)^{1/4}\log(1+u(x,t))|^2\,dx,\\ \int_{\mathbb{R}}|\partial_y v(y,\tau)|^2\,dy&=\int_{\mathbb{R}}|\partial_x\log(1+u(x,t))|^2(1+u(x,t))\,dx.\end{aligned}$$

Therefore, the results of the previous sections for (1.1) are translated to results for (8.1) as follows.

Theorem 8.1 *Let $v_0 \in L^1_+(\mathbb{R})$. There exists a unique global in time classical solution to equation* (8.1) *with initial value v_0.*

Theorem 8.2 *Let $v_0 \in L^1_+(\mathbb{R})$. The classical solution v to equation* (8.1) *with initial value v_0 satisfies:*

(i) L^1–L^∞ smoothing effect: $\|v(\cdot,\tau)\|_\infty \leq C\max\{\tau^{-1/2}\|v_0\|_1^{1/2}, \tau^{-3/4}\|v_0\|_1^{3/4}$ *for all* $\tau > 0$*;*

(ii) $\|v(\cdot,\tau)\|_1$ *and* $\|v(\cdot,\tau)\|_\infty$ *are non-increasing functions of* τ *in* $(0,\infty)$*;*

(iii) $\displaystyle\int_{\mathbb{R}} \left(1 - e^{-v(y,\tau)}\right)\,dy = \int_{\mathbb{R}} \left(1 - e^{-v_0(y)}\right)\,dy$ *for every* $\tau \geq 0$ (conservation law)*;*

(iv) $v \in C^{1,\alpha}(\mathbb{R}\times(0,\infty))$ *for every* $0 < \alpha < 1$*;*

(v) $v(y,\tau) > 0$ *for every* $y \in \mathbb{R}$, $\tau > 0$*;*

(vi) $v \in L^2_{\mathrm{loc}}((0,\infty) : H^1(\mathbb{R}))$.

Appendix A: A Nash-Trudinger inequality

In this appendix we contribute a new result that falls into the category of critical cases in embedding inequalities for spaces of functions with weak fractional derivatives.

Theorem A.1 *Let $\phi \in L^p(\mathbb{R}^N)$, $1 \leq p < \infty$, and assume that $(-\Delta)^{\gamma/2}\phi \in L^q(\mathbb{R}^N)$, $0 < \gamma < 1$, $q = N/\gamma$. Put $r = \max\{p, q\}$, $k = \lceil p/r' \rceil$, i.e., the least integer equal or larger than p/r', $r' = r/(r-1)$. There exists a constant $\alpha > 0$ such that if $\|\phi\|_p + \|(-\Delta)^{\gamma/2}\phi\|_q \leq 1$ then*

$$\int_{\mathbb{R}^N} \left(e^{\alpha|\phi|^{r'}} - \sum_{j=0}^{k-1} \frac{(\alpha|\phi|^{r'})^j}{j!} \right) \leq 1.$$

The particular case $p = q \leq 2$ was already proved by Strichartz in [27], using estimates on Bessel potentials. Note that in this case $k = 1$, so that the integrand is just $e^{\alpha|\phi|^{p'}} - 1$. Our result covers all the possibilities for the parameters in the critical case.

Before proceeding with the proof, we first review some related results and preliminaries for the reader's convenience.

Sobolev spaces of integer order. If $1 \leq p < N$, Sobolev's embedding shows that $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^r(\mathbb{R}^N)$ for all $p \leq r \leq Np/(N-p)$. If $p > N$, then $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^\infty(\mathbb{R}^N)$; even more, $\phi \in C^{0,1-N/p}(\mathbb{R}^N)$; the same happens for $p = N = 1$. The case $p = N > 1$ is critical and, though $W^{1,N}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for every $1 < N \le r < \infty$, it is easy to find examples of unbounded functions in $W^{1,N}(\mathbb{R}^N)$. However, if $p = N > 1$, then

$$\int_{\mathbb{R}^N} \left(e^{\alpha|\phi|^{\frac{N}{N-1}}} - \sum_{j=0}^{N-2} \frac{\left(\alpha|\phi|^{\frac{N}{N-1}}\right)^j}{j!} \right) \le 1 \tag{A.1}$$

for all ϕ in the unit ball of $W^{1,N}(\mathbb{R}^N)$, for some positive α independent of ϕ; see for example [1]. The proof of this result is based on the famous analogous estimate for the case of bounded domains due to Trudinger [29], later improved by Moser, [17].

Fractional Sobolev spaces. Let $1 \le q < \infty$, $0 < \gamma < 1$. The *homogeneous fractional Sobolev space* $\dot{W}^{\gamma,q}(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R})$ with the norm

$$\|\phi\|_{\dot{W}^{\gamma,q}} = \|(-\Delta)^{\gamma/2}\phi\|_q.$$

The standard fractional Sobolev spaces are defined trough the complete norm

$$\|\phi\|_{W^{\gamma,q}} = \|\phi\|_q + \|(-\Delta)^{\gamma/2}\phi\|_q.$$

The well-known Hardy-Littlewood-Sobolev inequality [14], [23], states that, if $q_* = Nq/(N-\gamma q)$, then

$$\|\phi\|_{q_*} \le C\|(-\Delta)^{\gamma/2}\phi\|_q,$$

for any $1 < q < N/\gamma$ and $0 < \gamma < 1$, and thus

$$\dot{W}^{\gamma,q}(\mathbb{R}^N) \subset L^{\frac{Nq}{N-\gamma q}}(\mathbb{R}^N). \tag{A.2}$$

Bessel potential spaces. The *Bessel potential spaces* are $L^{\gamma,q}(\mathbb{R}^N) = \{f = J_\gamma(\phi) : \phi \in L^q\}$, $1 \le q < \infty$, where J_γ (the Bessel potential of order $\gamma > 0$) is defined in terms of its Fourier transform,

$$\widehat{J_\gamma(f)}(\xi) = (1 + |\xi|^2)^{-\gamma/2}\widehat{f}(\xi).$$

These spaces were introduced by Aronszajn and Smith [2] and Calderón [7], and have a natural norm, $\|f\|_{L^{\gamma,q}} = \|J_{-\gamma}(f)\|_{L^q}$. The space $L^{\gamma,q}(\mathbb{R}^N)$ is equivalent to the above defined fractional Sobolev space $W^{\gamma,q}(\mathbb{R}^N)$ for every $0 < \gamma < 1$ and $1 \le q < \infty$; see Stein [24].

Critical Sobolev exponent. The inclusion (A.2) is not valid in the critical case $q = N/\gamma$, as pointed out in [26]. However, if in addition we know that $\phi \in L^q(\mathbb{R}^N)$, then $\phi \in L^r(\mathbb{R}^N)$ for $N/\gamma \le r < \infty$. That is, we have the inclusion

$$W^{\gamma,N/\gamma}(\mathbb{R}^N) \subset L^r(\mathbb{R}^N), \qquad \text{for every } N/\gamma \le r < \infty. \tag{A.3}$$

Indeed, in this situation ϕ belongs to the Bessel potential space $L^{\gamma,N/\gamma}(\mathbb{R}^N)$, and then the result follows from [27]. Notice that the case $r = \infty$ is not included; see [26].

To go beyond the L^r-spaces, $q \leq r < \infty$, in this critical case $q = N/\gamma$, a careful estimate of the norms of the inclusion (A.3) using estimates of the Bessel potentials, allowed Strichartz [27] to prove the inequality

$$\int_{\mathbb{R}^N} \left(e^{\alpha|\phi|^{\frac{N}{N-\gamma}}} - 1 \right) \leq 1 \tag{A.4}$$

for some $\alpha > 0$, valid for every ϕ such that $\|\phi\|_{L^{\gamma,N/\gamma}} \leq 1$. That is, $L^{\gamma,N/\gamma}(\mathbb{R}^N)$ is contained in the Orlicz space defined by the function in (A.4). But this result is restricted to the range $N/2 \leq \gamma < N$, unless the function ϕ has compact support. In our case of Sobolev spaces of fractional order $0 < \gamma < 1$, this means that only $N = 1$ can be considered, and then $1/2 \leq \gamma < 1$.

Besov spaces. On the other hand, Peetre [21] shows a restricted version of the previous inequality, in the spirit of (A.1), valid for every $0 < \gamma < N$, for functions in the Besov space $\Lambda_\gamma^{N/\gamma,N/\gamma}(\mathbb{R}^N)$; namely, there is a constant $\alpha > 0$ such that

$$\int_{\mathbb{R}^N} \left(e^{\alpha|\phi|^{\frac{N}{N-\gamma}}} - \sum_{j=0}^{k-1} \frac{(\alpha|\phi|^{\frac{N}{N-\gamma}})^j}{j!} \right) \leq 1, \tag{A.5}$$

$k = \lceil N/(N-\gamma) \rceil$, for every ϕ such that $\|\phi\|_{\Lambda_\gamma^{N/\gamma,N/\gamma}} \leq 1$.

The Besov spaces $\Lambda_\gamma^{p,q}(\mathbb{R}^N)$ are defined through the norm

$$\|\phi\|_{\Lambda_\gamma^{p,q}} = \|\phi\|_q + \left(\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^q}{|x-y|^{N+\gamma q}} \, dx \right)^{p/q} dy \right)^{1/p};$$

see [5]. It turns out that $\Lambda_\gamma^{2,2}(\mathbb{R}^N) = W^{\gamma,2}(\mathbb{R}^N)$. However, Besov spaces with $p = q$ and Sobolev spaces are different whenever $q \neq 2$ [25].

In the above-mentioned results, the control of both the function and its derivatives, or some quantity related to the derivatives, in the *same* L^p space, yields a control in some Orlicz space. Our aim in Theorem A.1 is to show an Orlicz-type estimate analogous to (A.1) and (A.5) starting from a control of the function and its derivatives in *different* L^p spaces.

We first obtain a generalization of the critical Sobolev-type embedding (A.3),

Proposition A.1 *Let $p \geq 1$ and $0 < \gamma < 1$ we have*

$$L^p(\mathbb{R}^N) \cap \dot{W}^{\gamma,N/\gamma}(\mathbb{R}^N) \subset L^r(\mathbb{R}^N) \quad \textit{for every } p \leq r < \infty.$$

Proof. It follows from the Nash-Gagliardo-Nirenberg type inequality

$$\|\phi\|_{rs}^r \leq C(q,\gamma,N)p\|(-\Delta)^{\gamma/2}\phi\|_q \|\phi\|_p^{r-1}, \quad r = p + 1 - p/q, \; s = N/(N-\gamma),$$

valid for any function $\phi \in L^p(\mathbb{R}^N) \cap \dot{W}^{\gamma,q}(\mathbb{R}^N)$, $p \geq 1$, $q > 1$, $0 < \gamma < 1$, proved by the authors in [20]. Indeed, in the particular case $q = N/\gamma$ we have $s = q'$, and thus

$$\|\phi\|_{p+q'}^{p+q'} \leq Cp^{q'} \|(-\Delta)^{\gamma/2}\phi\|_q^{q'} \|\phi\|_p^p. \tag{A.6}$$

$\square$

Proof of Theorem A.1. As mentioned before, the particular case $p = q \leq 2$ was already proved in [27]. We will show how to treat the rest of the cases to get a complete analysis.

Case $p = q > 2$. It was also covered in [27] under the additional restriction of asking ϕ to be compactly supported. For general functions some easy modification is needed. Indeed, for any function $\phi \in L^{\gamma,q}(\mathbb{R}^N)$, $q = N/\gamma$, such that $\|\phi\|_{L^{\gamma,q}} \leq 1$, the following estimate holds

$$\|\phi\|_{L^r} \leq A\left(1 + \frac{r}{q'}\right)^{1/r+1/q'}$$

for every $q \leq r < \infty$, where the constant A depends on N and q, but not on r, see [27]. We take then $r = jq'$, $j \geq q-1$ (which implies $r \geq q$), and obtain

$$\sum_{j\geq q-1} \frac{c^j \|\phi\|_{jq'}^{jq'}}{j!} \leq \sum_{j\geq -1} \frac{c^j A^{jq'}(j+1)^{j+1}}{j!} < \infty$$

if we choose $c > 0$ small enough. Finally, in order to have 1 in the right-hand side of (A.1), we use that the function

$$F(t) = e^{t^{q'}} - \sum_{j=0}^{k-1} \frac{t^{jq'}}{j!}$$

satisfies $F(\lambda t) \leq \lambda^{q'} F(t)$ for every $t > 0$, $0 < \lambda < 1$.

Case $p < q$. Using the Nash-Gagliardo-Nirenberg type inequality (A.6) we conclude that $\phi \in L^q(\mathbb{R}^N)$, and thus $\phi \in L^{\gamma,q}(\mathbb{R}^N)$. We apply then the previous case.

Case $p > q$. The key idea is that there is a value $0 < \mu < \gamma$ such that $\phi \in L^{\mu,N/\mu}(\mathbb{R}^N)$. Indeed, we can reach the exponent of integration p by lowering the order of differentiation. This follows from the Hardy-Littlewood-Sobolev inclusion (A.2), which can be written as

$$\dot{W}^{\gamma_2,N/\gamma_2}(\mathbb{R}^N) \subset \dot{W}^{\gamma_1,N/\gamma_1}(\mathbb{R}^N) \qquad \text{for every } 0 < \gamma_1 < \gamma_2.$$

Hence, for the precise choice $\mu = N/p$ we obtain that $(-\Delta)^{\mu/2}\phi \in L^p(\mathbb{R}^N)$. We may now apply the case $p = q$ with γ replaced by $\mu = N/p$. $\square$

Remark. One is tempted to use the Nash-Gagliardo-Nirenberg inequality (A.6) in order to estimate the sum in the development of the function in (A.1). Unfortunately, the coefficient in (A.6) makes the sum divergent.

Appendix B: A calculus inequality

In the course of the proof of the smoothing effect we use a nice calculus inequality. Since it is not evident, we include a proof for the sake of completeness.

Lemma B.1 *For every $x, a \geq 0$ we have*

$$(e^{ax}-1)^2 \leq (e^a-1)(e^{ax^2}-1)\,.$$

Proof. We develop the function $f(x) = (e^a-1)(e^{ax^2}-1) - (e^{ax}-1)^2$ in its Taylor series and rearrange the terms as follows:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{a^n}{n!} \sum_{k=1}^{\infty} \frac{x^{2k}a^k}{k!} - \Big(\sum_{n=1}^{\infty} \frac{x^n a^n}{n!}\Big)^2 \\ &= \sum_{n\neq k}^{\infty} \frac{1}{n!\,k!}(a^{n+k}x^{2k} - a^{n+k}x^{n+k}) \\ &= \sum_{n\neq k}^{\infty} \frac{1}{n!\,k!} a^{n+k}x^{n+k}(x^{k-n}-1)\,. \end{aligned}$$

By grouping the twin terms (n,k) and (k,n) we may restrict ourselves to the cases $k > n$ and then

$$f(x) = \sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} \frac{1}{n!\,k!} a^{n+k}x^{n+k}(x^{k-n} + x^{n-k} - 2) \geq 0,$$

since the last factor is always positive for $x \neq 1$ and vanishes for $x = 1$. □

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References

[1] Adams, R. A.; Fournier, J. J. F. "Sobolev spaces". Second edition. Pure and Applied Mathematics (Amsterdam), 140. Elsevier/Academic Press, Amsterdam, 2003. ISBN: 0-12-044143-8.

[2] Aronszajn, N.; Smith, K. T. *Theory of Bessel potentials. I.* Ann. Inst. Fourier (Grenoble) 11 (1961) 385–475.

[3] Athanasopoulos, I.; Caffarelli, L. A. *Continuity of the temperature in boundary heat control problems.* Adv. Math. 224 (2010), no. 1, 293–315.

[4] Bénilan, P.; Gariepy, R. *Strong solutions in L^1 of degenerate parabolic equations.* J. Differential Equations 119 (1995), no. 2, 473–502.

[5] Besov, O. V. *Investigation of a class of function spaces in connection with imbedding and extension theorems.* (Russian) Trudy. Mat. Inst. Steklov. 60 (1961), 42–81.

[6] Caffarelli, L. A.; Vasseur, A. *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation.* Ann. of Math. (2) 171 (2010), no. 3, 1903–1930.

[7] Calderón, A.-P. *Lebesgue spaces of differentiable functions and distributions.* 1961, Proc. Sympos. Pure Math., Vol. IV, pp. 33–49, American Mathematical Society, Providence, R.I.

[8] Cifani, S.; Jakobsen, E. R. *Entropy solution theory for fractional degenerate convection-diffusion equations*. Ann. Inst. H. Poincaré Anal. Non Linéaire 28 (2011), no. 3, 413–441.

[9] Constantin, P.; Lax, P.; Majda, A. *A simple one-dimensional model for the three-dimensional vorticity.* Comm. Pure Appl. Math. 38 (1985), no. 6, 715–724.

[10] Córdoba, A.; Córdoba, D.; Fontelos, M. A. *Formation of singularities for a transport equation with nonlocal velocity.* Ann. of Math. (2) 162 (2005), no. 3, 1377–1389.

[11] Crandall, M. G. *An introduction to evolution governed by accretive operators*. In "Dynamical systems" (Proc. Internat. Sympos., Brown Univ., Providence, R.I., 1974), Vol. I, pp. 131–165. Academic Press, New York, 1976.

[12] Crandall, M. G.; Liggett, T. M. *Generation of semi-groups of nonlinear transformations on general Banach spaces*. Amer. J. Math. 93 (1971), 265–298.

[13] Dong, H. *Well-posedness for a transport equation with nonlocal velocity.* J. Funct. Anal. 255 (2008), no. 11, 3070–3097.

[14] Hardy, G. H.; Littlewood, J. E. *Some properties of fractional integrals. I.* Math. Z. 27 (1928), no. 1, 565–606.

[15] Kiselev, A.; Nazarov, F.; Shterenberg, R. *Blow up and regularity for fractal Burgers equation*. Dyn. Partial Differ. Equ. 5 (2008), no. 3, 211–240.

[16] Landkof, N. S. "Foundations of modern potential theory". Die Grundlehren der mathematischen Wissenschaften, Band 180. Springer-Verlag, New York-Heidelberg, 1972.

[17] Moser, J. *A sharp form of an inequality by N. Trudinger*. Indiana Univ. Math. J. 20 (1970/71), 1077–1092.

[18] Oleinik, O. A.; Kalashnikov, A. S.; Czou, Y.-I. *The Cauchy problem and boundary problems for equations of the type of non-stationary filtration*. Izv. Akad. Nauk SSSR. Ser. Mat. 22 (1958), 667–704. (Russian).

[19] de Pablo, A.; Quirós, F.; Rodríguez, A.; Vázquez, J. L. *A fractional porous medium equation.* Adv. Math. 226 (2011), no. 2, 1378–1409.

[20] de Pablo, A.; Quirós, F.; Rodríguez, A.; Vázquez, J. L. *A general fractional porous medium equation.* Comm. Pure Appl. Math. 65 (2012), no. 9, 1242–1284.

[21] Peetre, J. *Espaces d'interpolation et théorème de Soboleff.* (French) Ann. Inst. Fourier (Grenoble) 16 (1966), fasc. 1, 279–317.

[22] Silvestre, L. *Regularity of the obstacle problem for a fractional power of the Laplace operator.* Comm. Pure Appl. Math. 60 (2007), no. 1, 67–112.

[23] Sobolev, S. L. *On a theorem of functional analysis.* Transl. Amer. Math. Soc. 34(2) (1963), 39–68; translation of Mat. Sb. 4 (1938) 471–497.

[24] Stein, E. M. *The characterization of functions arising as potentials.* Bull. Amer. Math. Soc. 67 (1961) 102–104.

[25] Stein, E. M. "Singular integrals and differentiability properties of functions", Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970.

[26] Strichartz, R. S. *Multipliers on fractional Sobolev spaces.* J. Math. Mech. 16 (1967) 1031–1060.

[27] Strichartz, R. S. *A note on Trudinger's extension of Sobolev's inequalities.* Indiana Univ. Math. J. 21 (1971/72), 841–842.

[28] Stroock, D. W. "An introduction to the theory of large deviations". Universitext. Springer-Verlag, New York, 1984. ISBN: 0-387-96021-X.

[29] Trudinger, N. S. *On imbeddings into Orlicz spaces and some applications.* J. Math. Mech. 17 (1967), 473–483.

[30] Varopoulos, N. Th. *Hardy-Littlewood theory for semigroups.* J. Funct. Anal. 63 (1985), no. 2, 240–260.

[31] Vázquez, J. L. "Smoothing and decay estimates for nonlinear diffusion equations. Equations of porous medium type". Oxford Lecture Series in Mathematics and its Applications, 33. Oxford University Press, Oxford, 2006. ISBN: 978-0-19-920297-3; 0-19-920297-4.

[32] Vázquez, J. L. "The porous medium equation. Mathematical theory". Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007. ISBN: 978-0-19-856903-9.